Chapter 31

The Contextual Computer

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What kind of computer is the universe? Here we present three results. The first is a consequence of the Kochen-Specker theorem: If the predictions of quantum mechanics are correct, then the universe cannot be a non-contextual computer. We then show that, if we assume that the density of memory is bounded, then the universe cannot be a classical contextual computer. The third result singles out the universe among all possible contextual computers by exploiting a curious connection with graph theory: In the universe, the maximal contextuality of a set of propositions is given by the Lovász number of the graph representing their mutual exclusiveness.

1. Introduction

Physicists have convincing arguments explaining why the universe cannot be a classical computer working on bits. The most famous one is Bell’s theorem,\(^1\) which can be formulated in a very general way: If the predictions of quantum mechanics are correct and if the speed of information is limited, then no classical computer can simulate the results we obtain in some experiments.

But let us suppose that our experiments are never fast enough to guarantee that they cannot influence the results generated in other parts of the universe. If such is the case, is there any way to prove that the universe cannot be a classical computer?

In Sec. 2 we define the concept of non-contextual computer and, in Sec. 3, we show that the universe cannot be a non-contextual computer. This is a well-known result which, by itself, does not prevent the universe to be a
2. Contextuality

Throughout all this article, we will consider the following scenario: A single portion of the universe (a “system”) is subjected to a sequence of dichotomic experiments (that is, with only two possible results: +1 or −1). The experiments will be chosen from amongst a finite set. The order in which the experiments are performed and the number of times each experiment is performed is randomly decided. The goal is to design a computer which simulates the results of these experiments, assuming that these results are correctly predicted by quantum mechanics.

We choose the experiments to show one of the fundamental properties of quantum mechanics: Contextuality. To introduce contextuality, we first need to introduce two concepts: Repeatability and compatibility.

An experiment $A$ is repeatable when it can be performed many (ideally infinite) times and gives always the same result. We assume that no other experiment is performed in between two experiments $A$. In quantum mechanics, a repeatable experiment is called a sharp measurement or a projective measurement and is mathematically represented by a self-adjoint matrix (a square matrix with complex entries that is equal to its own conjugate transpose).

Two repeatable experiments $A$ and $B$ are compatible when every time experiment $B$ is performed after experiment $A$, a subsequent execution of experiment $A$ yields the same result as if experiment $B$ had not been performed. Compatibility is an experimentally testable property. It is symmetric (the roles of $A$ and $B$ are interchangeable), but not necessarily transitive: Even if $A$ is compatible with $B$, and $B$ is compatible with $b$, it may happen that $A$ and $b$ are incompatible. A set of experiments $S = \{A, B, \ldots, Z\}$ is compatible when all experiments in $S$ are mutually compatible. In quantum mechanics, compatible experiments are represented by commuting matrices, $AB = BA$. In classical physics, carefully performed experiments are always compatible. This is not the case in quantum mechanics.

We focus on a particular type of incompatible experiments. Two repeatable dichotomic experiments $A$ and $b$ are maximally incompatible if, whenever $b$ is performed after $A$, a subsequent execution of $A$ yields a fundamentally unpredictable result. This implies that it yields the same result as if $b$ had not been performed with probability $\frac{1}{2}$, and yields the opposite result with probability $\frac{1}{2}$. Maximal incompatibility is also an experimentally testable, symmetric but not necessarily transitive property. In quantum mechanics, maximally incompatible experiments are represented by anticommuting matrices, $Ab = -bA$.

A non-contextual computer is any which provides pre-established results which are not affected by compatible experiments. For instance, suppose experiment $A$ is compatible with experiments $B$ and $a$ (although $B$ and $a$ may be mutually incompatible). The assumption of non-contextuality is that the result of $A$ is the same, regardless of whether $A$ is performed alone, after $B$, or after $a$. However, the result of $A$ might be different if an incompatible experiment $b$ were to be performed before $A$. A contextual computer is any which does not satisfy the definition of non-contextual computer.

We assume that the system on which these experiments are performed is a set of $n$ qubits (with $n \geq 2$). A qubit is any physical system for which we have the ability of preparing and measuring all the possible quantum superpositions of two perfectly distinguishing states. For example, a qubit is the polarization of a single photon. In contrast, a classical bit is any physical system which we can prepare either in state 0, in state 1, or in a probabilistic mixture of them, and on which we can distinguish states 0 and 1. A qubit is therefore more general than a bit and much more expensive. A dead/alive cat is a bit, but not a qubit unless we can prepare any possible quantum superpositions and we perform any possible quantum sharp measurement of dead and alive.

3. The Impossible Non-Contextual Computer

The goal is to design a computer which simulates some predictions of quantum mechanics. Consider 9 dichotomic experiments $A, B, C, a, b, c, \alpha, \beta$ and $\gamma$, such that they form 6 compatible sets: $S_1 = \{A, B, C\}$, $S_2 = \{a, b, c\}$, $S_3 = \{\alpha, \beta, \gamma\}$, $S_4 = \{a, \alpha, c\}$, $S_5 = \{B, b, \beta\}$ and $S_6 = \{C, c, \gamma\}$, and such
that any pair of experiments is either compatible or maximally incompatible. The predictions of quantum mechanics we want to simulate are the following: (i) The compatible/maximally incompatible experiments are actually compatible/maximally incompatible. (ii) The product of the results of the three experiments in $S_i$ performed sequentially is $+1$ for $i = 1, \ldots, 5$ and $-1$ for $i = 6$. That is, now using $A$ to denote the result of experiment $A$, etc.,

$$ABC = 1,$$
$$abc = 1,$$
$$\alpha \beta \gamma = 1,$$
$$A = 1,$$
$$Bb = 1,$$

$$C = 1.$$ (1)

A non-contextual computer should provide results for each of the 9 experiments which do not depend on which other compatible experiments are performed.

The interesting point is that a non-contextual computer cannot reproduce all the predictions in (1). The proof is simple: Assuming a non-contextual assignment of results, if we multiply the first three equations in (1), we obtain

$$ABCabca\beta \gamma = 1.$$ (2)

However, if we multiply the last three equations in (1), we obtain

$$ABCabca\beta \gamma = -1.$$ (3)

The contradiction proves that the assumption is not valid: Non-contextual assignments are impossible.

One way to produce the quantum predictions in (1) is to pick $n = 2$ qubits and choose the following experiments: $A = \sigma_x \otimes I$ (meaning the measurement of the observable represented in quantum mechanics by the tensor product of the Pauli matrix $x$ and the $2 \times 2$ identity matrix), $B = \sigma_y \otimes I$, $C = \sigma_z \otimes I$, $a = I \otimes \sigma_x$, $b = \sigma_y \otimes I$, $c = \sigma_z \otimes I$, $\alpha = \sigma_x \otimes \sigma_y$, $\beta = \sigma_y \otimes \sigma_x$, and $\gamma = \sigma_z \otimes \sigma_x$. This is a simple example of a general result in quantum mechanics known as the Kochen-Specker theorem.\footnote{Reference number.}

4. A Classical Contextual Computer Requires Unlimited Density of Memory

There is a simple solution to the task of simulating predictions (i) and (ii): The computer saves in its memory the result of the first measurement, e.g., $A$. If the second measurement is maximally incompatible, e.g., $b$, then the computer erases from its memory the result of $A$ and saves the result of $b$. If the second measurement is compatible with $A$, e.g., $B$, then the computer saves both the results of $A$ and $B$. The first equation in (1) automatically defines the result of $C = AB$. In this way, the computer can provide contextual results simulating (i) and (ii). All the computer needs is memory. In principle, not too much memory: All that the computer needs to keep is the number of states needed to remember which was the set of compatible experiments $S_i$ of the last compatible measurements (e.g., $S_1$) and the number of states needed to remember the results of all the experiments in $S_i$ (e.g., $A$, $B$ and $C$).

For example, to reproduce (i) and (ii) in the previous scenario, the computer needs $\log_2(6 \times 2^2) \approx 4.58$ bits, since there are 6 possible sets $S_i$ and $2^2$ possible states in every $S_i$ (the result of the third experiment is defined by the results of the other two). A formal proof that this is the necessary and sufficient memory needed to satisfy (i) and (ii), assuming that the result of an experiment incompatible with the previous one is genuinely random, can be found in.\footnote{Reference number.} Notice that this memory is already larger than the information-carrying capacity of two qubits, which is two bits.\footnote{Reference number.} The computer needs to store more information than that which is accessible by performing experiments on the computer. Strange, but, so far, not too bad.

Let us add all possible experiments which are either compatible or maximally incompatible with the previous experiments. The new experiments are: $\sigma_x \otimes I$, $\sigma_y \otimes I$, $\sigma_z \otimes I$, $\sigma_x \otimes \sigma_z$, $\sigma_y \otimes \sigma_z$, $\sigma_z \otimes \sigma_y$ and $\sigma_y \otimes \sigma_z$. We end up with the set $E^{(2)} = \{ E^{(2)}_i \}_{i=1}^{15}$ of 15 experiments which can be defined for $n = 2$ qubits using the tensor product of one of the three Pauli matrices or the identity matrix times one of the three Pauli matrices or the identity matrix (we exclude $I \otimes I$ in the count, because it is compatible with all the other experiments and not maximally incompatible with any of them). $E^{(2)}$ defines a set $S^{(2)} = \{ S^{(2)}_j \}_{j=1}^{15}$ of 15 sets $S^{(2)}_j$ of three compatible experiments $E^{(2)}_i$. We will denote by $S^{(2)}$ the pair $(E^{(2)}, S^{(2)})$.

$S^{(2)}$ can be arranged in 10 different sets of 9 experiments and 6 compatible sets, each defining an equation like those in (1) for which no non-
contextual model exists. Since no non-contextual model exists for each of these 10 subsets, then no non-contextual model exits for $S^{(2)}$.

The addition of the 6 extra experiments had the purpose of completing the set of experiments on $n = 2$ qubits in which every pair is either compatible or maximally incompatible. $S^{(2)}$ presents an advantage compared to the initial set of experiments in Sec. 3: The ratio between the quantum predictions that cannot be satisfied with non-contextual models and the total number of quantum predictions increases from $\frac{1}{6}$ for the set in Sec. 3 to $\frac{1}{3}$ for $S^{(2)}$. Therefore, to simulate (i) and (ii) for $S^{(2)}$, the computer needs more memory. To be precise,

$$m(2) = \log_2(15 \times 2^2) \approx 5.91 \text{ bits.} \quad (4)$$

Another good thing about $S^{(2)}$ is that it can be naturally generalized to any $n \geq 2$: For a system of $n \geq 2$ qubits, a complete set of experiments such that each pair is either compatible or maximally incompatible is the one consisting of all $n$-fold tensor products of $\sigma_x$, $\sigma_y$, $\sigma_z$ and $\mathbb{I}$. We will call this set $S^{(n)} = (E^{(n)}, S^{(n)})$. From a practical point of view, a useful property of $S^{(n)}$ is that performing the experiments $E^{(n)}_i$ for arbitrary $n$ only requires assembling the devices needed for the experiments in $S^{(2)}$ (for a particular example, see\textsuperscript{11}).

$E^{(n)}$ has $4^n - 1$ experiments $E^{(n)}_i$. $S^{(n)}$ has

$$c(n) = \frac{1}{n(n + 1)} \prod_{k=1}^{n} (2^k + 1) \prod_{k=0}^{n-1} (2^n - 2^k), \quad (5)$$

compatible sets $S_j^{(n)}$ containing a maximum number of elements.\textsuperscript{9} Each $S_j^{(n)}$ contains $2^n - 1$ mutually compatible experiments, but only the results of $2^n - 2$ of them are independent, since the product of all of them is either +1 or -1, according to quantum mechanics. Therefore, each $S_j^{(n)}$ can be in

$$s(n) = 2^{(2^n - 2)} \quad (6)$$

different states.

Non-contextual models cannot simulate all these quantum predictions. Indeed, the fraction of predictions which cannot be satisfied by a non-contextual model increases rapidly with $n$: it is $\frac{3}{15} = \frac{1}{5} = 0.2$ for $n = 2$, $\frac{4}{18} = \frac{4}{9} \approx 0.27$ for $n = 3$ and $\frac{157248}{385560} = \frac{104}{255} \approx 0.41$ for $n = 4$.\textsuperscript{10}

The proof in\textsuperscript{6} can be easily extended to arbitrary $n$. This means that the memory needed to simulate (i) and (ii) is the one needed to remember which compatible set $S_j^{(n)}$ the last experiments belongs to, and which are the results for all the independent elements in that set. This means that, for arbitrary $n \geq 2$, the memory is

$$m(n) = \log_2 [c(n) \times s(n)], \quad (7)$$

which is larger than $2^n$ for $2 \leq n \leq 29$; my computer produces “overflow” for $n \geq 30$. Therefore, the memory required to reproduce predictions (i) and (ii) grows at least exponentially with the number of qubits. In other words, the density of bits of memory per qubit required to simulate the results as pre-established contextual properties also grows exponentially with $n$. This means that, if there is a limitation to the density of memory a finite computer can have, e.g., 1 bit per qubit or $10^6$ bits per qubit, at some point this limitation will make it impossible to simulate the predictions of quantum mechanics (assuming an unlimited number of qubits in the universe).

Even more appealing than the possibility of a computer with unlimited density of memory is to think that the information is not in the $n$-qubit system, but in the observer, and that the system is only carrying $n$ bits of information.

5. The Contextuality of the Universe

Now we know what kind of computer the universe is unlikely to be (a non-contextual or a classical one), but apparently nothing has been said about what kind of computer the universe is. However, we have learnt something which is of fundamental importance: The universe is a contextual computer such that the precise amount of contextuality in every situation can be obtained from quantum mechanics. The only missing piece is which is the fundamental principle responsible for this exact amount of contextuality. We still do not have the answer, but we have made some progress.

Let us first explain how to quantify contextuality. Consider the predictions in (1). We know that they are satisfied by quantum mechanics but not by non-contextual models. This can be expressed by saying that, if

$$\kappa := P(ABC = 1) + P(abc = 1) + P(\alpha \beta \gamma = 1)$$

$$+ P(Aaa = 1) + P(Bb\beta = 1) + P(Cc\gamma = -1), \quad (8)$$

where $P(ABC = 1)$ is the probability of obtaining results for $A$, $B$ and $C$ such that $ABC = 1$, then $\kappa \leq 5$ for non-contextual theories, while $\kappa \leq 6$ for quantum mechanics.
Now let us assume that \( C = c = \alpha = -\beta = \gamma = i\varphi \) in (8). Then, we can define

\[
\kappa' := P(AB = 1) + P(ab = 1) + P(Aa = 1) + P(Bb = -1). \tag{9}
\]

What are the bounds for \( \kappa' \)? The answer is \( \kappa' \leq 3 \) for non-contextual theories and \( \kappa' \leq 2 + \sqrt{2} \approx 3.41 \) for quantum mechanics. The first bound can be achieved by choosing 4 mutually compatible experiments \( A, B, a \) and \( b \) and preparing a state such that \( A = B = a = b = +1 \). The second, by preparing two qubits in the state \( |\psi\rangle = \frac{1}{\sqrt{2}}|\sigma_z = +1\rangle \otimes |\sigma_z = -1\rangle \) and performing the experiments \( A, B, a \) and \( b \) described in Sec. 3. How to prove that no higher values are reachable? There is a simple method. Since \( P(AB = 1) = P(A = B = +1) + P(A = B = -1) \), then \( \kappa' \) can be expressed as

\[
\kappa' := P(A = B = +1) + P(A = B = -1) \\
+ P(a = b = +1) + P(a = b = -1) \\
+ P(A = a = +1) + P(A = a = -1) \\
+ P(B = -b = +1) + P(B = -b = -1). \tag{10}
\]

In any non-contextual model, every system must have precise values for \( A, B, a \) and \( b \). Therefore, if the proposition \( "A = B = +1" \) is true for a specific system, then the propositions \( "A = B = -1", "A = a = -1" \) and \( "B = -b = -1" \) must be false. What is the maximum number of true values we can assign to the 8 propositions in \( \kappa' \)? As a simple inspection reveals, the answer is 3. And the same limit holds no matter how we choose the non-contextual values. Therefore, since the value of \( \kappa' \) is obtained by performing experiments on different systems, the highest value \( \kappa' \) can take is necessarily 3. The same method explains why \( \kappa \leq 5 \) for non-contextual theories.

Quantum mechanics is different. In quantum mechanics, the maximum value is

\[
\vartheta(G) := \max \sum_{i=1}^{p} |\langle \psi|v_i\rangle|^2, \tag{11}
\]

where the maximum is taken over all unit vectors \( |\psi\rangle \) and \( |v_i\rangle \), in any dimension, where each \( |v_i\rangle \) corresponds to a proposition in \( \kappa' \) (or \( \kappa \)) and exclusive propositions are represented by orthogonal vectors. \( p \) is the number of explicit propositions: \( p = 6 \times 2^2 = 24 \) in \( \kappa \) ("\( A = B = -C = +1 \)", "\( A = -B = -C = +1 \), \ldots, "\( C = c = \gamma = -1 \)"), and \( p = 4 \times 2 = 8 \) in \( \kappa' \).

However, the most curious thing is that both numbers have been used for a long time in graph theory. Let us construct a graph \( G \) in which vertices represent the propositions and edges connect those that cannot both be true. For example, the graph associated to \( \kappa \) is shown in Fig. 1, and the graph associated to \( \kappa' \) is a subgraph of this graph. Then, the non-contextual bound is the independence number of \( G \), \( \alpha(G) \), and the quantum bound is the Lovász number of \( G \), \( \vartheta(G) \). This was first observed in.\(^{14} \) Remarkably, while computing the non-contextual maximum is NP-hard, the quantum one (the one which represents what happens in the universe) can be computed to arbitrary precision by semidefinite programming in polynomial time.\(^{13} \) The question of whether this number naturally derives from some fundamental principle is still open.

Fig. 1. Graph \( G \) representing all the explicit propositions in \( \kappa \) defined in (9). Each vertex represents a proposition and adjacent vertices represent propositions that cannot both be true. True/false assignments are represented by green/red-shaped circles. For \( G \), the maximum number of propositions which can be true in a non-contextual model is \( \alpha(G) = 5 \). However, the maximum quantum value for \( \kappa \) is \( \vartheta(G) = 6 \).
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References


Chapter 32

A Gödel-Turing Perspective on Quantum States Indistinguishable from Inside*

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By a diagonalisation argument, Bell states are not distinguishable from inside. This result is closely related to the theorems of Gödel, Church, and Turing in spite of important dissimilarities.

1. Introduction

Consider two entangled quantum systems $A, B$. Without loss of generality take $A$ and $B$ to be qubits. $A$ and $B$ are allowed to communicate on a classical channel the results of measurements performed by each of them. Can $A$ or $B$, by themselves or in cooperation, immediately or with hindsight, discriminate between two different entangled states of the joint system $A\&B$? We argue that they cannot discriminate states $\rho_1, \rho_2$ of $A\&B$ whose partial trace over both $A$ and $B$ coincide, i.e. for which $\text{tr}_A(\rho_1) = \text{tr}_A(\rho_2)$ and $\text{tr}_B(\rho_1) = \text{tr}_B(\rho_2)$. These conditions are satisfied for example for the density matrices of the Bell states

$$\rho_1 = (|00\rangle \langle 00| + |11\rangle \langle 00| + |00\rangle \langle 11| + |11\rangle \langle 11|)/2$$

$$\rho_2 = (|00\rangle \langle 11| - |11\rangle \langle 00| - |00\rangle \langle 11| + |11\rangle \langle 11|)/2$$

(There are two more Bell states, which we will not use here. Our indistinguishability results are valid for any two of the four Bell states, and in fact for any two quantum states differing only in the entanglement between $A$ and $B$.)

These internal indistinguishability results differ from other well-known restrictions of the distinguishability of quantum states. One such argument establishes that there is no quantum procedure to reliably distinguish non-orthogonal states, see e.g. Nielsen and Chuang [1, p.87]. Holevo\(^2\)

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