NOVEL BELL INEQUALITIES FOR n QUBITS DISTRIBUTED BETWEEN m < n PARTIES

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We describe a method for obtaining m-partite Bell inequalities that are maximally violated by n-qubit states by an amount that grows exponentially with n (n > m). These inequalities, derived for states with perfect correlations, are, however, valid for all local hidden variable theories.

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1. Introduction

Local hidden variable (LHV) theories are those in which all local observables possess definite outcomes. Bell inequalities are constraints satisfied by any LHV theory on the values of linear combinations of averages (or probabilities) of products of outcomes of sufficiently separated experiments on distant physical systems. Local experiments are “sufficiently separated” when the outcome of each local experiment is spacelike separated from the choice(s) of the corresponding local experiment(s) on the distant system(s).

Bell inequalities have played a fundamental role in quantum information. Questions like which quantum states violate Bell inequalities,1 or what are the consequences of the failure of Einstein, Podolsky, and Rosen’s (EPR’s) criterion for elements of reality2 in a cryptographic scenario3 marked the birth of the entanglement theory and the entanglement-based secure key distribution, respectively. The range of topics in which Bell inequalities have had a decisive role is impressive, from multipartite entanglement4 to communication complexity.5 Bell inequalities have become a common technique as entanglement witnesses6 and as a means for guarantee the security of a key distribution with untrusted devices.7

Here we describe a method for obtaining a novel type of m-partite Bell inequalities valid for any LHV theory. These Bell inequalities applies to a scenario where n qubits are distributed between m < n parties, so that party i receives $n_i$ qubits
A. Cabello

The ingredients for this method have been introduced separately in several papers. This is the first attempt to provide a comprehensive view.

One motivation for these Bell inequalities is the recent possibility of using the same physical system as a carrier of several qubits (e.g. the same photon can carry a qubit in its polarization and other qubit in its linear momentum). Then, a single detection of the carrier gives information that would require several detections in a scenario where each physical system carries only one qubit. This constitutes an advantage for physical systems, like photons, where the detection loophole is a serious obstacle for a conclusive (loophole-free) Bell experiment.

Other motivation is the fact that the resulting m-partite Bell inequalities are violated by an amount which grows exponentially with n, even though the number of parties would remain low. This is also of interest to elude the detection loophole, since, at least for some quantum states, the minimum required detection efficiency for a conclusive Bell experiment is a monotone function of the amount of violation.

These two properties can eventually help to design a photonic loophole-free Bell experiment, since the required minimum detection efficiency for some of these new inequalities is expected to be below the 0.67 value required for the most promising proposal for a photonic loophole-free Bell experiment to date.

2. Description of the Method

Now we will describe how to obtain a m-party n-qubit Bell inequality by means of an example with m = 2 parties and n = 4 qubits. Although the resulting Bell inequality is valid for any LHV theory, the method is based on the properties of a specific quantum state that will provide the maximum violation of the inequality. Starting with a different state, we will end with a different Bell inequality. The method has 3 steps.

2.1. Optimal n-partite Bell inequality for a n-qubit graph state

The method starts by choosing a particular n-qubit graph state. For instance, the 4-qubit cluster state

$$|LC_4\rangle = \frac{1}{2}(-|0000\rangle - |0011\rangle - |1100\rangle + |1111\rangle),$$

where $Z|0\rangle = |0\rangle$, $Z|1\rangle = -|1\rangle$, $X|0\rangle = |1\rangle$, and $X|1\rangle = -|0\rangle$, being $Z$ and $X$ the corresponding Pauli matrices. This state is the unique state satisfying

$$g_i|LC_4\rangle = |LC_4\rangle, \quad i = 1, 2, 3, 4,$$

where

$$g_1 = X_1Z_2,$$
$$g_2 = Z_1X_2Z_3,$$

$$g_3 = X_1Z_2X_3,$$
$$g_4 = Z_1X_2Z_3X_4.$$
where, e.g. $Z_j$ is the Pauli $Z$ matrix of qubit $j$.

This state violates a Bell inequality where the Bell operator has the $2^n$ terms of the stabilizer of the state (i.e. contains all the perfect correlations of the state).

In our example, this Bell operator is

$$\mathcal{B} = 1 + g_1 + g_2 + g_3 + g_4 + g_1 g_2 + \cdots + g_3 g_4 + g_1 g_2 g_3 + \cdots + g_2 g_3 g_4 + g_1 g_2 g_3 g_4,$$

where $1$ denotes the identity matrix. However, $\mathcal{B}$ is not the Bell operator providing the maximum violation. As mentioned before, we are interested in the Bell operator giving the maximum violation because this usually means that this is the one requiring the lowest minimum detection efficiency for a loophole-free Bell experiment. For the $|LC_4\rangle$, the Bell operator containing only perfect correlations and giving the maximum violation is

$$\beta = (1 + g_1)g_2(1 + g_3)$$

$$= g_2 + g_2 g_3 + g_1 g_2 + g_1 g_2 g_3$$

$$= Z_1 X_2 Z_3 + Z_1 Y_2 Y_3 Z_4 + Y_1 Y_2 Z_3 - Y_1 X_2 Y_3 Z_4.$$  \hspace{1cm} (5)

Note that $\beta$ is contained in $\mathcal{B}$. Assuming that $Z_1$, $Y_1$, $X_2$, $Y_2$, $Z_3$, $Y_3$, and $Z_4$ have definite outcomes, $-1$ or $1$, it is easy to see that the following Bell inequality holds:

$$\langle Z_1 X_2 Z_3 \rangle + \langle Z_1 Y_2 Y_3 Z_4 \rangle + \langle Y_1 Y_2 Z_3 \rangle - \langle Y_1 X_2 Y_3 Z_4 \rangle \leq 2.$$  \hspace{1cm} (6)

However, for the $|LC_4\rangle$, the expected value of the Bell operator $\beta$ is

$$\langle LC_4 | \beta | LC_4 \rangle = 4,$$  \hspace{1cm} (7)

because

$$Z_1 X_2 Z_3 |LC_4\rangle = |LC_4\rangle,$$  \hspace{1cm} (8a)

$$Z_1 Y_2 Y_3 Z_4 |LC_4\rangle = |LC_4\rangle,$$  \hspace{1cm} (8b)

$$Y_1 Y_2 Z_3 |LC_4\rangle = |LC_4\rangle,$$  \hspace{1cm} (8c)

$$-Y_1 X_2 Y_3 Z_4 |LC_4\rangle = |LC_4\rangle.$$  \hspace{1cm} (8d)

The 4-partite Bell inequality (6) was first derived in Ref. 26. It has been recently proved that it is optimal in the sense mentioned above. Reference 12 also contains all optimal Bell inequalities for each and every graph state of $n \leq 6$ qubits. The 4-partite Bell inequality (6) has been tested experimentally using 4 photons entangled in polarization.
2.2. \textit{m}-partite Bell inequality for a \textit{n}-qubit graph state

The Bell inequality resulting from the previous step is a \textit{n}-partite Bell inequality, where \textit{n} is the number of qubits of the state (i.e. each party has one qubit). We can make the assumption that in any LHV theory \(Z_1, Y_1, X_2, Y_2, Z_3, Y_3, \) and \(Z_4\) have definite outcomes, \(-1\) or \(1\), because, e.g., the outcome of measuring \(Z_1\) on qubit 1 can be spacelike separated from the choices of measurements on qubits 2, 3, and 4.

If we distribute these \(n = 4\) qubits between \(m = 2\) parties \(A\) and \(B\), e.g. both qubits 1 and 2 (3 and 4) are carried by particle \(A\) (\(B\)), then we cannot assume that in any LHV theory, e.g. \(Z_1\) has a definite outcome regardless of whether we decide to measure \(X_2\) or \(Y_2\) on qubit 2, because this decision cannot be spacelike separated from the outcome \(Z_1\) and therefore can, in principle, influence the outcome \(Z_1\).

However, there is a scenario involving 2 (or 3) parties where we can assume that \(Z_1, Y_1, X_2, Y_2, Z_3, Y_3, \) and \(Z_4\) have definite outcomes, \(-1\) or \(1\), by invoking EPR's criterion for elements of reality. EPR's criterion states that: “if, without in any way-disturbing a system, we can predict with certainty (i.e. with probability equal to unity) the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity.”\(^2\) Applied to the \(m = 2\) case, this means that we can assume that \(Z_1, Y_1, X_2, Y_2, Z_3, Y_3, \) and \(Z_4\) are elements of reality (i.e. have definite outcomes, \(-1\) or \(1\)), if it is possible to predict with certainty the outcomes of the measurements on particle \(A\) using only the outcomes of spacelike separated chosen measurements on particle \(B\), and vice versa.

This trick was first described in Refs. 29 and 30 in a slightly different context, and was first applied to the \(|LC_4\rangle\) in Ref. 8. If particle \(A\) carries qubits 1 and 4, and particle \(B\) carries qubits 2 and 3, then we can assume that \(Z_1, Y_1, X_2, Y_2, Z_3, Y_3, \) and \(Z_4\) have definite outcomes because, for the \(|LC_4\rangle\), the outcomes of these single-qubit measurements on \(A\) (\(B\)) can be predicted with certainty from the outcomes of single-qubit measurements on \(B\) (\(A\)). Specifically, in the \(|LC_4\rangle\), these outcomes must satisfy:

\begin{align*}
Z_1 &= X_2 Z_3, & (9a) \\
Y_1 &= Y_2 Z_3, & (9b) \\
X_2 &= Z_1 X_4, & (9c) \\
Y_2 &= Y_1 X_4, & (9d) \\
Z_3 &= X_4, & (9e) \\
Y_3 &= X_1 Y_4, & (9f) \\
Z_4 &= Z_2 X_3, & (9g)
\end{align*}

since

\begin{align*}
&\langle LC_4|Z_1X_2Z_3|LC_4\rangle = 1, & (10a) \\
&\langle LC_4|Y_1Y_2Z_3|LC_4\rangle = 1, & (10b)
\end{align*}
\[ \langle LC_4|Z_1X_2X_4|LC_4 \rangle = 1, \quad (10c) \]
\[ \langle LC_4|Y_1Y_2X_4|LC_4 \rangle = 1, \quad (10d) \]
\[ \langle LC_4|Z_3X_4|LC_4 \rangle = 1, \quad (10e) \]
\[ \langle LC_4|X_1Y_4|LC_4 \rangle = 1, \quad (10f) \]
\[ \langle LC_4|Z_2X_3Z_4|LC_4 \rangle = 1. \quad (10g) \]

Indeed, the \(|LC_4\rangle\) is the only graph state with less than 6 qubits that admits a distribution of the qubits between 2 particles in which the EPR criterion can be invoked to assume definite outcomes for the Pauli matrices.\(^{11}\)

The important consequence of all this is that the 4-partite Bell inequality (6) can be converted into the following 2-partite Bell inequality:

\[ \langle Z_A^1X_B^2Z_C^3 \rangle + \langle Z_A^1Z_B^2Y_D^3 \rangle + \langle Y_A^1Y_B^2Z_C^3 \rangle - \langle Y_A^1Z_B^2X_C^2Y_D^3 \rangle \leq 2, \quad (11) \]

where, e.g. \(Z_A^1X_B^2Z_C^3\) refers to an experiment in which \(Z_A^1\) is measured on particle \(A\), and \(X_B^2\) and \(Z_C^3\) are measured on particle \(B\). Note that a single detection of particle \(B\) is enough to reveal the outcomes of \(X_B^2\) and \(Z_C^3\).

The 2-partite Bell inequality (11) has been tested experimentally by using a 2-photon \(|LC_4\rangle\), where the photons are entangled in polarization and linear momentum.\(^{31}\)

All possible \(n\)-qubit graph states (with \(n \leq 7\)) and distributions of qubits between \(m = 2\) parties allowing 2-partite Bell inequalities can be found in Ref. 11. Given a graph state, the problem of finding the minimum \(m\) allowing a \(m\)-partite Bell inequality is solved in Ref. 14.

2.3. \(m\)-partite Bell inequality for any \(n\)-qubit state

There is, however, a problem with the 2-partite Bell inequality (11): it is valid only for LHV theories in which all the conditions (9a)–(9g) hold. That is, it is valid if we can prepare a perfect 2-photon \(|LC_4\rangle\). However, although actual preparations have good fidelity, they are not perfect.\(^{31}\) Here we describe how to deal with this problem. We will transform the 2-partite Bell inequality (11), which is valid only for LHV theories satisfying (9a)–(9g), into a 2-partite Bell inequality valid for any LHV theory. This method was also presented in Ref. 13 in a slightly different context.

Suppose that our preparation is such that

\[ \langle Z_1^AX_2^BX_3^C \rangle = 1 - \epsilon_1, \quad (12a) \]
\[ \langle Y_1^AY_2^BY_3^C \rangle = 1 - \epsilon_2, \quad (12b) \]
\[ \langle Z_1^AX_2^BX_3^C \rangle = 1 - \epsilon_3, \quad (12c) \]
\[ \langle Y_1^AY_2^BX_3^C \rangle = 1 - \epsilon_4, \quad (12d) \]
\[ \langle X_2^BY_3^C \rangle = 1 - \epsilon_5, \quad (12e) \]
where $0 \leq \epsilon_i \leq 1$, and $\epsilon_i = 0$ indicates perfect correlation and $\epsilon_i = 1$ indicates that there is no correlation. For any large ensemble of pairs, $\epsilon_i$ can be obtained experimentally. Therefore, the ensemble of pairs of particles can be divided into, e.g. a fraction $1 - (\epsilon_1/2)$ of pairs for which $\langle Z_1^A X_2^B Z_3^B \rangle = 1$, and a fraction $\epsilon_1/2$ of pairs for which $\langle Z_1^A X_2^B Z_3^B \rangle = -1$. For this fraction the 2-partite Bell inequality (11) is not a legitimate Bell inequality. However, at least for a fraction $1 - \sum_{i=1}^{7} \epsilon_i/2$ of the pairs, the 2-partite Bell inequality (11) is a legitimate Bell inequality. The remaining fraction $\sum_{i=1}^{7} \epsilon_i/2$ of pairs must satisfy the trivial inequality

$$\langle Z_1^A X_2^B Z_3^B \rangle + \langle Z_1^A Z_4^A Y_2^B Y_3^B \rangle + \langle Y_1^A Y_2^B Z_3^B \rangle - \langle Y_1^A Z_4^A X_2^B Y_3^B \rangle \leq 4.$$  (13)

Adding (11) and (13) with weights $1 - \sum_{i=1}^{7} \epsilon_i/2$ and $\sum_{i=1}^{7} \epsilon_i/2$, respectively, yields,

$$\langle Z_1^A X_2^B Z_3^B \rangle + \langle Z_1^A Z_4^A Y_2^B Y_3^B \rangle + \langle Y_1^A Y_2^B Z_3^B \rangle - \langle Y_1^A Z_4^A X_2^B Y_3^B \rangle \leq 2 + \sum_{i=1}^{7} \epsilon_i.$$  (14)

This 2-partite Bell inequality is valid for any LHV theory and can therefore be tested without making any assumption regarding the preparation.

3. Open Questions and Future Developments

The main open question of this approach is what is the minimum detection efficiency required for loophole-free violation of a $m$-party $n$-qubit Bell inequality like (14). Even for simple cases, calculating this detection efficiency is a difficult task. The utility of these Bell inequalities for the design of a loophole-free Bell experiment depends on the solution of this problem.

On the other hand, we expect to have 2- and 4-photon 6- and 8-qubit states in the near-future. There, the use of these novel $m$-partite Bell inequalities can lead to the largest experimental violations of a Bell inequality to date.
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