

LETTERS

State-independent experimental test of quantum contextuality

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The question of whether quantum phenomena can be explained by classical models with hidden variables is the subject of a long-lasting debate^{1,2}. In 1964, Bell showed that certain types of classical models cannot explain the quantum mechanical predictions for specific states of distant particles, and some types of hidden variable models^{3–9} have been experimentally ruled out. An intuitive feature of classical models is non-contextuality: the property that any measurement has a value independent of other compatible measurements being carried out at the same time. However, a theorem derived by Kochen, Specker and Bell^{10–12} shows that non-contextuality is in conflict with quantum mechanics. The conflict resides in the structure of the theory and is independent of the properties of special states. It has been debated whether the Kochen–Specker theorem could be experimentally tested at all^{13,14}. First tests of quantum contextuality have been proposed only recently, and undertaken with photons^{15,16} and neutrons^{17,18}. But these tests required the generation of special quantum states and left various loopholes open. Here we perform an experiment with trapped ions that demonstrates a state-independent conflict with non-contextuality. The experiment is not subject to the detection loophole and we show that, despite imperfections and possible measurement disturbances, our results cannot be explained in non-contextual terms.

Hidden variable models assert that the result $v(A)$ of measuring the observable A on an individual quantum system is predetermined by a hidden variable λ . Two observables A and B are mutually compatible if the result of A does not depend on whether B is measured before, after or simultaneously with A and vice versa. Non-contextuality is the property of a hidden variable model that the value $v(A)$ is predetermined regardless of which other compatible observable is measured jointly with A . Kochen and Specker showed that the assumption of non-contextuality cannot be reconciled with quantum mechanics. A considerable simplification of the original Kochen–Specker argument by Mermin and Peres^{19,20} uses a 3×3 square of observables A_{ij} with possible outcomes $v(A_{ij}) = \pm 1$, where the observables in each row or column are mutually compatible. Considering the products of rows $R_k = v(A_{k1})v(A_{k2})v(A_{k3})$ and columns $C_k = v(A_{1k})v(A_{2k})v(A_{3k})$, the total product would be $\prod_{k=1,2,3} R_k C_k = 1$, as any $v(A_{ij})$ appears twice in the total product.

In quantum mechanics, however, one can take a four-level quantum system, for instance two spin- $\frac{1}{2}$ -particles, and the following array of observables:

$$\begin{array}{lll} A_{11} = \sigma_z^{(1)} & A_{12} = \sigma_z^{(2)} & A_{13} = \sigma_z^{(1)} \otimes \sigma_z^{(2)} \\ A_{21} = \sigma_x^{(2)} & A_{22} = \sigma_x^{(1)} & A_{23} = \sigma_x^{(1)} \otimes \sigma_x^{(2)} \\ A_{31} = \sigma_z^{(1)} \otimes \sigma_x^{(2)} & A_{32} = \sigma_x^{(1)} \otimes \sigma_z^{(2)} & A_{33} = \sigma_y^{(1)} \otimes \sigma_y^{(2)} \end{array} \quad (1)$$

Here $\sigma_i^{(k)}$ denotes the Pauli matrix acting on the k th particle, and all the observables have the outcomes ± 1 . Moreover, in each of the rows

or columns of the Mermin–Peres square (1), the observables are mutually commuting and can be measured simultaneously or in any order, hence they are compatible. In any row or column, their measurement product R_k or C_k equals 1, except for the third column where it equals -1 . Therefore, quantum mechanics yields for the product $\prod_{k=1,2,3} R_k C_k$ a value of -1 , in contrast to non-contextual models.

To test this property, it has to be expressed as an inequality because no experiment yields ideal quantum measurements. Recently, it has been shown that the inequality

$$\langle \mathcal{X}_{KS} \rangle = \langle R_1 \rangle + \langle R_2 \rangle + \langle R_3 \rangle + \langle C_1 \rangle + \langle C_2 \rangle - \langle C_3 \rangle \leq 4 \quad (2)$$

holds for all non-contextual theories²¹, where $\langle \dots \rangle$ denotes the ensemble average. Quantum mechanics predicts for any state that the mean value in the Kochen–Specker inequality is $\langle \mathcal{X}_{KS} \rangle = 6$, thereby violating inequality (2). For an experimental test, an ensemble of quantum states Ψ needs to be prepared and each realization subjected to the measurement of one of the possible sets of compatible observables. Here, it is of utmost importance that all measurements of A_{ij} are context-independent²¹, that is, A_{ij} must be detected with a quantum non-demolition (QND) measurement that provides no information whatsoever about any other compatible observable.

Experiments processing quantum information with trapped ions²² are particularly well-suited for this purpose, as arbitrary two-qubit quantum states Ψ can be deterministically generated by laser–ion interactions and measured with near-unit efficiency⁶. Two ionic energy levels are defined to represent the qubit basis states $|\uparrow\rangle$ and $|\downarrow\rangle$, which are eigenstates of the observable σ_z . The qubit is measured by electron shelving²² projecting onto $|\uparrow\rangle$ or $|\downarrow\rangle$. Measurement of any other observable A_{ij} is reduced to detecting $\sigma_z^{(k)}$, $k = 1$ or 2 , by applying a suitable unitary transformation U to the state Ψ before measuring $\sigma_z^{(k)}$, and its inverse $U^\dagger$ after the measurement (see Fig. 1 and Methods). With these basic tools, any set of observables can be sequentially measured in an arbitrary temporal order.

For the experiment, a pair of $^{40}\text{Ca}^+$ ions is trapped in a linear Paul trap with axial and radial vibrational frequencies of $\omega_{\text{ax}} = (2\pi) 1.465$ MHz and $\omega_r \approx (2\pi) 3.4$ MHz, respectively, and Doppler-cooled by exciting the $S_{1/2} \leftrightarrow P_{1/2}$ and $P_{1/2} \leftrightarrow D_{3/2}$ dipole transitions. Optical pumping initializes an ion with a fidelity of 99.5% to the qubit state $|\downarrow\rangle \equiv |S_{1/2}, m = 1/2\rangle$, the second qubit state being $|\uparrow\rangle \equiv |D_{5/2}, m = 3/2\rangle$. The qubit is coherently manipulated²³ by an ultrastable, narrow-band laser coherently exciting the $S_{1/2} \leftrightarrow D_{5/2}$ quadrupole transition in a magnetic field of $B = 4$ G. Single-qubit light-shift gates $U_z^{(1)}(\theta) = \exp(-i\frac{\theta}{2}\sigma_z^{(1)})$ are realized by an off-resonant beam impinging on ion 1 with a beam waist of $3\ \mu\text{m}$ and a k -vector perpendicular to the ion string. A second beam, illuminating both ions

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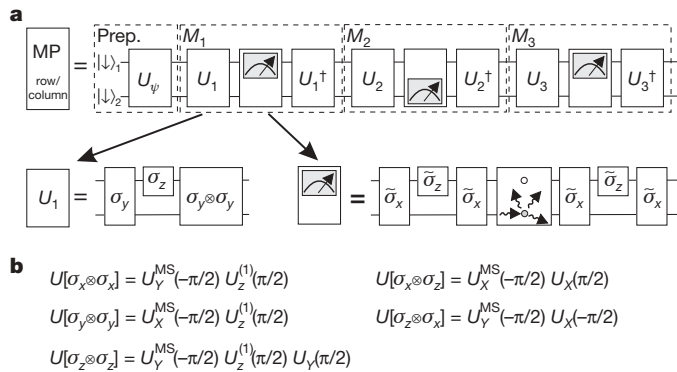


Figure 1 | Experimental measurement scheme. **a**, For the measurement of the j th row (column) of the Mermin–Peres (MP) square (1), a quantum state is prepared (Prep.) on which three consecutive QND measurements M_k , $k = 1, 2, 3$, are performed measuring the observables $A_{jk}(A_{kj})$. Each measurement consists of a composite unitary operation U_k that maps the observable of interest onto one of the single-qubit observables $\sigma_z^{(1)}$ or $\sigma_z^{(2)}$, which are measured by fluorescence detection. The unitary operations U are synthesized from single-qubit and maximally entangling gates. In the lower row, $\sigma_i, \bar{\sigma}_i$ symbolize, respectively, the Hamiltonian acting on the qubit and on the $D_{5/2}$ Zeeman subspace spanned by $\{|\downarrow\downarrow\rangle, |a\rangle\}$ which is used for hiding one ion's quantum state from fluorescence light during detection³⁰. **b**, All mapping operations U_k used for measuring the five two-qubit spin correlations $\sigma_i^{(1)} \otimes \sigma_j^{(2)}$ require an entangling gate. Here, we list the gate decompositions of $U[\sigma_i^{(1)} \otimes \sigma_j^{(2)}]$ used in the experiments, where $U_X(\theta) = U(\theta, \phi = 0)$ and $U_Y(\theta) = U(\theta, \phi = \pi/2)$.

with equal strength at an angle of 45° with respect to the ion crystal, serves to carry out gate operations that are symmetric under qubit exchange. Collective single-qubit gates $U(\theta, \phi) = \exp(-i\frac{\theta}{2}(\sigma_\phi^{(1)} + \sigma_\phi^{(2)}))$, where $\sigma_\phi^{(k)} = \cos(\phi)\sigma_x^{(k)} + \sin(\phi)\sigma_y^{(k)}$, are realized by resonantly exciting the qubit transition and controlling the phase ϕ of the laser light. If instead a bichromatic light field near-resonant with the upper and lower sideband transitions of the axial centre-of-mass (COM) mode is used, a Mølmer–Sørensen gate²⁴ $U^{\text{MS}}(\theta, \phi) = \exp(-i\frac{\theta}{2}\sigma_\phi^{(1)} \otimes \sigma_\phi^{(2)})$ is implemented^{23,25}. We achieve a maximally entangling gate ($\theta = \pi/2$) capable of mapping $|\downarrow\downarrow\rangle$ to $|\downarrow\downarrow\rangle + ie^{-i2\phi}|\uparrow\uparrow\rangle$ with a fidelity of about 98%, even for Doppler-cooled ions in a thermal state with an average of $\bar{n}_{\text{ax,COM}} \approx 18$ vibrational quanta. This property is of crucial importance, as the experiment demands gate operations subsequent to quantum state detection by fluorescence measurements which do not preserve the motional quantum number. The set of elementary gates²⁶ $\{U_z^{(1)}(\theta), U(\theta, \phi), U^{\text{MS}}(\theta, \phi)\}$ is sufficient to construct the two-qubit unitary operations needed for creating various input states Ψ and mapping the observables A_{ij} to $\sigma_z^{(k)}$ for read-out (see Methods).

Equipped with these tools, we create the singlet state $\Psi = (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)/\sqrt{2}$ by applying the gates $U_z^{(1)}(\pi)U(\frac{\pi}{2}, \frac{3\pi}{4})U^{\text{MS}}(\frac{\pi}{2}, 0)$ to the initial state $|\downarrow\downarrow\rangle$ and measure consecutively the three observables of a row or column of the Mermin–Peres square. The results obtained for a total of 6,600 copies of Ψ are visualized in Fig. 2. The three upper panels (Fig. 2a–c) show the distribution of measurement results $\{v(A_{11}), v(A_{12}), v(A_{13})\}$ and their products for the observables appearing in the rows of the Mermin–Peres square (1), and the three lower panels (Fig. 2d–f) show the corresponding results for the columns of the square. Figure 2f demonstrates that Ψ is a common eigenstate of the observables $\sigma_x^{(1)} \otimes \sigma_x^{(2)}, \sigma_y^{(1)} \otimes \sigma_y^{(2)}, \sigma_z^{(1)} \otimes \sigma_z^{(2)}$, as only one of the spheres has a considerable size. Five of the correlations have a value close to +1 whereas $\langle C_3 \rangle = -0.91(1)$. By adding them up and subtracting $\langle C_3 \rangle$, we find a value of $\langle \chi_{\text{KS}} \rangle = 5.46(4) > 4$, thus violating equation (2). Here and elsewhere in the paper, errors are due to slowly fluctuating experimental control parameters.

To test the prediction of a state-independent violation, we repeated the experiment for nine other quantum states of different

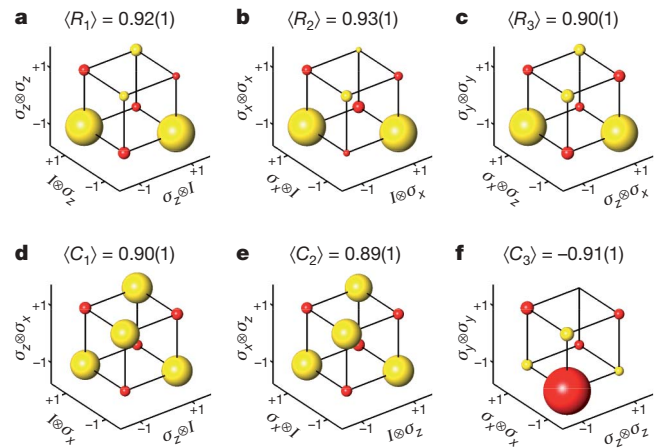


Figure 2 | Measurement correlations for the singlet state. **a**, Visualization of the consecutive measurement of the three observables $A_{11} = \sigma_z^{(1)}$, $A_{12} = \sigma_z^{(2)}$, $A_{13} = \sigma_z^{(1)} \otimes \sigma_z^{(2)}$ corresponding to row 1 of the Mermin–Peres square. The measurement is carried out on 1,100 preparations of the singlet state. The volume of the spheres on each corner of the cube represents the relative frequency of finding the measurement outcome $\{v_1, v_2, v_3\}$, $v_i \in \{\pm 1\}$. The colour of the sphere indicates whether $v_1v_2v_3 = +1$ (yellow) or -1 (red). The average of the measurement product $\langle R_1 \rangle$ is given at the top. **b–f**, Measurements of the remaining rows (**b, c**) or columns (**d–f**) of the Mermin–Peres square. Errors given are 1σ .

purity and entanglement. Figure 3 shows that a state-independent violation of the Kochen–Specker inequality indeed occurs, with $\langle \chi_{\text{KS}} \rangle$ ranging from 5.23(5) to 5.46(4). We also checked that a violation of inequality (2) occurs irrespective of the temporal order of the measurement triples. Figure 4 shows the results for all possible permutations of the rows and columns of (1), based on 39,600 realizations of the singlet state. When combining the correlation results for the 36 possible permutations of operator orderings in (1), we find that all values for the Kochen–Specker inequality $\langle \chi_{\text{KS}} \rangle$ fall within the range from 5.22 to 5.49. Because of experimental imperfections, the experimental violation of the Kochen–Specker inequality falls short of the quantum-mechanical prediction. The dominating error

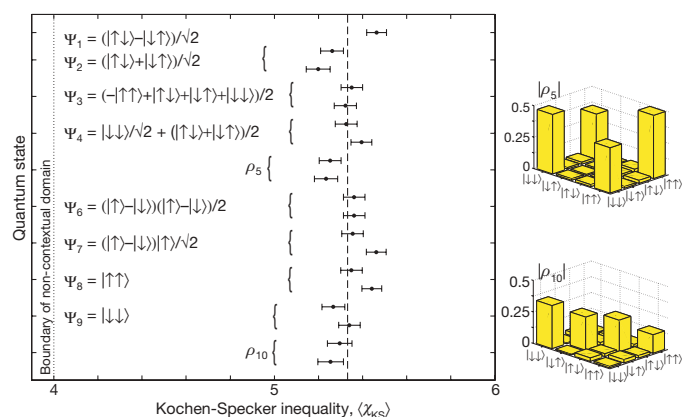


Figure 3 | State-independence of the Kochen–Specker inequality. The Kochen–Specker inequality was tested for ten different quantum states, including maximally entangled (Ψ_1 – Ψ_3), partially entangled (Ψ_4) and separable (Ψ_6 – Ψ_9) almost pure states, as well as an entangled mixed state (ρ_5) and an almost completely mixed state (ρ_{10}). All states are analysed by quantum state tomography, which yields for the experimentally produced states Ψ_1 – Ψ_4, Ψ_6 – Ψ_9 an average fidelity of 97(2)%. For ρ_5 and ρ_{10} , the tomographic reconstruction is shown at right. For all states, we obtain a violation of inequality (2) which demonstrates its state-independent character, the vertical dashed line indicating the average value of $\langle \chi_{\text{KS}} \rangle$. Error bars, 1σ (6,600 state realizations per data point).

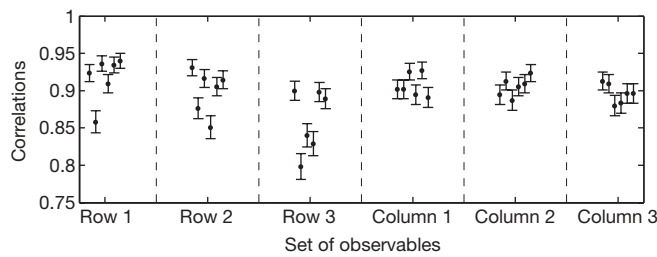


Figure 4 | Permutation within rows and columns of the Mermin–Peres square. As the three observables of a set are commuting, the temporal order of their measurements should have no influence on the measurement results. The figure shows the measured absolute values of the products of observables for any of the six possible permutations. The scatter in the experimental data is caused by experimental imperfections that affect different permutations differently. For the measurements shown here, 39,600 copies of the singlet state were used in total. Error bars, 1σ .

sources are imperfect unitary operations, in particular the entangling gates applied up to six times in a single experimental run.

All experimental tests of hidden-variable theories are subject to various possible loopholes. In our experiment, the detection loophole does not have a role, as the states of the ions are detected with near-perfect efficiency. From the point of view of a hidden variable theory, objections can still be made: in the experiment, owing to imperfect mapping operations, the observables are not perfectly compatible and as the observables are measured sequentially, it may be that the hidden variables are disturbed during the sequence of measurements, weakening the necessity to assign to any observable a fixed value independently of the context.

Nevertheless, it is possible to derive inequalities for classical non-contextual models, wherein the hidden variables are disturbed during the measurement process (see Methods). More specifically, it can be proved that the probabilities of measurement outcomes for the disturbed hidden variables then obey the inequality:

$$\begin{aligned} \langle \mathcal{X}_{\text{DHV}} \rangle &= \langle A_{12}A_{13} \rangle + \langle A_{22}A_{23} \rangle + \langle A_{12}A_{22} \rangle - \langle A_{13}A_{23} \rangle \\ &\quad - 2p^{\text{err}}[A_{13}A_{12}A_{13}] - 2p^{\text{err}}[A_{23}A_{22}A_{23}] \\ &\quad - 2p^{\text{err}}[A_{22}A_{12}A_{22}] - 2p^{\text{err}}[A_{23}A_{13}A_{23}] \leq 2 \end{aligned} \quad (3)$$

Here, $\langle A_i A_j \rangle = \langle \nu(A_i) \nu(A_j) \rangle$ denotes the ensemble average if A_i is measured before A_j and $p^{\text{err}}[A_i A_j A_i]$ denotes the probability that measuring A_j introduces a change in the value of A_i if the sequence $A_i A_j A_i$ is measured. To test this inequality, we prepared the state $\Psi \propto |\uparrow\uparrow\rangle + i|\uparrow\downarrow\rangle + \gamma|\downarrow\uparrow\rangle + i|\downarrow\downarrow\rangle$ where $\gamma = \sqrt{2} - 1$, and measured the Mermin–Peres square with σ_y and σ_z exchanged. We find for the whole square the value $\langle \mathcal{X}_{\text{KS}} \rangle = 5.22(10) > 4$, and for equation (3) the value $\langle \mathcal{X}_{\text{DHV}} \rangle = 2.23(5) > 2$. This proves that even disturbances of the hidden variables during not perfectly compatible measurements cannot explain the given experimental data.

In principle, our analysis of measurement disturbances and dynamical hidden variable models can be extended to the full Mermin–Peres square, but the experimental techniques have to be improved to find a violation there. Our findings already show unambiguously that the experimentally observed phenomena cannot be described by non-contextual models. Remarkably, the experimental observation here of counter-intuitive quantum predictions did not require the preparation of specific entangled states with non-local correlations. We expect this result to stimulate new applications of quantum contextuality for quantum information processing^{27–29}.

METHODS SUMMARY

In quantum physics experiments, there is often a preferred basis for carrying out measurements. In the case of trapped ion qubits, measurements projecting onto eigenstates of σ_z are easily carried out by a fluorescence detection that couples only one of the qubit states $|\uparrow\rangle$, $|\downarrow\rangle$ to an excited state that rapidly decays back to the same qubit state. In the two-ion experiment described in this paper, the

observables of interest, $\sigma_i^{(1)} \equiv \sigma_i \otimes I$, $\sigma_j^{(2)} \equiv I \otimes \sigma_j$ and $\sigma_i^{(1)} \otimes \sigma_j^{(2)}$, all have two doubly-degenerate eigenvalues. Therefore, these observables can be measured by unitarily mapping them to $\sigma_i^{(k)}$, $k = 1$ or 2 , measuring $\sigma_i^{(k)}$ and applying the inverse mapping operation. For spin correlations $\sigma_i \otimes \sigma_j$, the mapping requires the application of entangling two-qubit gates.

The inequality (3) holds for hidden variable models, which assign definite values to any observable and allow the probability distribution of the hidden variables to change during sequential measurements. It is assumed that (on average) a measurement of an observable A followed by a measurement of an observable B disturbs the predetermined value of A more strongly than a measurement of B alone. Then, from a Clauser–Horne–Shimony–Holt-type inequality²¹, the inequality (3) follows.

Full Methods and any associated references are available in the online version of the paper at www.nature.com/nature.

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Author Contributions G.K., F.Z. and R.G. performed the experiment and partially analysed the data; M.K., O.G. and A.C. provided the theoretical part and the modelling of imperfect measurements; C.F.R. conceived the experiment and analysed the data; R.B., G.K., F.Z., R.G. and C.F.R. contributed to the experimental set-up; and all authors co-wrote the paper.

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METHODS

Quantum state detection. The quantum state of a single ion is detected by illuminating both ions with light near-resonant with the $S_{1/2} \leftrightarrow P_{1/2}$ and $P_{1/2} \leftrightarrow D_{3/2}$ transitions for 250 μs . To prevent the quantum information of the other ion from being read out, the quantum state of the qubit that is not to be detected is hidden in the $D_{5/2}$ -Zeeman state manifold by a composite π -pulse transferring the $|\downarrow\rangle$ state's population to the auxiliary state $|a\rangle \equiv |D_{5/2}, m=5/2\rangle$ before fluorescence detection. After the detection, the qubit is restored in the state space $\{|\uparrow\rangle, |\downarrow\rangle\}$ by another composite π -pulse. The parameters of the read-out lasers are set such that it keeps the axial COM-mode in the Lamb-Dicke regime²² with $\bar{n}_{\text{COM}} \approx 18$. By combining the counts of two photomultipliers, we observe a Poissonian distribution of photon counts in the detection window with average count numbers $n_{\uparrow} \approx 7.8$ and $n_{\downarrow} \approx 0.07$ in the bright and dark state, respectively. Setting the discrimination threshold to 1.5 counts, the conditional probabilities for wrong quantum state assignments amount to $p(\uparrow|\downarrow) \approx 0.24\%$ and $p(\downarrow|\uparrow) \approx 0.39\%$. At the end of the detection interval, the ion is optically pumped on the $S_{1/2} \leftrightarrow P_{1/2}$ to prevent leakage of population from the qubit level $|\downarrow\rangle$ to the state $|S_{1/2}, m=-1/2\rangle$.

QND measurements of spin correlations. Quantum non-demolition measurements of observables A_{ij} that measure spin correlations are carried out by mapping the subspace $\mathcal{H}_{A_{ij}}^+ = \{\psi | A_{ij}\psi = \psi\}$ onto the subspace $\mathcal{H}_{\sigma_z^{(2)}}^+ = \{\psi | \sigma_z^{(2)}\psi = \psi\}$ before the fluorescence measurement of $\sigma_z^{(2)}$. This is achieved by applying a unitary state transformation U_{ij} satisfying $A_{ij} = U_{ij}^\dagger \sigma_z^{(2)} U_{ij}$ to the two-qubit state of interest. To decompose U_{ij} into the elementary gate operations available in our set-up, we use a gradient-ascent based numerical search routine²⁶. After measurement of $\sigma_z^{(2)}$, the inverse operation $U_{ij}^\dagger$ completes the QND measurement.

Modelling imperfect measurements. To deal with the case of imperfect measurements from a hidden variable viewpoint, let us assume that there is a hidden variable model that determines jointly the probabilities of the results of all sequences of measurements. Such probabilities are written as $p[A^{(1)+}, B^{(2)-}; AB]$ and so on, denoting the probability for the result $A^{(1)} = +1$ and $B^{(2)} = -1$ when measuring first A and then B . Then the probabilities fulfil:

$$p\left[\left(A^{(1)+}; A\right) \text{ and } \left(B^{(1)+}; B\right)\right] \leq p\left[A^{(1)+}, B^{(2)+}; AB\right] + p\left[\left(B^{(1)+}; B\right) \text{ and } \left(B^{(2)-}; AB\right)\right] \quad (4)$$

This inequality holds because if the hidden variable λ is such that it contributes to $p[A^{(1)+}; A]$ and $p[B^{(1)+}; B]$, then either the value of B stays the same when measuring A and λ contributes to $p[A^{(1)+}, B^{(2)+}; AB]$, or it is flipped and it contributes to $p[B^{(1)+}; B]$ and $p[B^{(2)-}, AB]$. The term $p[A^{(1)+}, B^{(2)+}; AB]$ is directly measurable, and we estimate the other term by assuming:

$$p\left[\left(B^{(1)+}; B\right) \text{ and } \left(B^{(2)-}; AB\right)\right] \leq p\left[B^{(1)+}, B^{(3)-}; BAB\right] \quad (5)$$

This inequality means that the disturbance of a predetermined value of B caused by the measurement of B and A should be not smaller than the disturbance due to measurement of A alone, as the former includes additional experimental procedures compared with the latter. The probability $p[B^{(1)+}, B^{(3)-}; BAB]$ is experimentally accessible, and combining equation (4) and (5) we obtain a measurable upper bound on $p[A^{(1)+}; A]$ and $p[B^{(1)+}; B]$. Then, the Clauser-Horne-Shimony-Holt-type inequality²¹ $\langle AB \rangle + \langle CD \rangle + \langle AC \rangle - \langle BD \rangle \leq 2$ holds, where the correlations like $\langle AB \rangle$ are obtained from the probabilities $p[A^{(1)+}; A]$ and $p[B^{(1)+}; B]$ and so on. Using the above bounds and the notation $p^{\text{err}}[BAB] = p[B^{(1)+}, B^{(3)-}; BAB] + p[B^{(1)-}, B^{(3)+}; BAB]$, one arrives at the inequality (3) for sequences of measurements.

The probabilities $p^{\text{err}}[BAB]$ quantify the compatibility of the implemented observables. For the measurements in inequality (3) they are found to be 0.047(17) on average. For the full Mermin-Peres square, a similar analysis would need error probabilities of the kind $p^{\text{err}}[BAB]$ and $p^{\text{err}}[CABC]$ to deal, for example, with the term $\langle ABC \rangle$. For a measured value of $\langle \mathcal{X}_{\text{KS}} \rangle = 5.46$, they would have to be on average smaller than 0.03 in order to find a violation of an inequality similar to (3). This error level, however, is more difficult to achieve as measurement results have to be correlated that involve subsequent measurement of up to four observables.