1. INTRODUCTION

Bell’s theorem states that quantum mechanics cannot be reproduced by any “local realistic theory” [1]. A Bell inequality is a constraint imposed by these theories on the values of a linear combination $\beta$ of the averages of the results of experiments on two or more separated systems. However, Bell’s original inequality [1] is not a Bell inequality in the modern sense [2]. The premise of the original Bell inequality is not locality, as in Clauser-Horne-Shimony-Holt (CHSH) type Bell inequalities [3,4], but Einstein, Podolsky and Rosen’s (EPR’s) criterion for the existence of elements of reality [5], which establishes two conditions for the existence of elements of reality: (i) Perfect predictability: it must be possible to predict them with certainty. (ii) Locality: the prediction must be based on a measurement that exerts no disturbing influence upon them. Bell’s original inequality is based on the fact that, for the two-qubit singlet state, the results of measuring the same observable on both qubits are perfectly correlated: Bell’s original inequality is based on an equality.

The interesting point is this inequality is the first of a family of Bell inequalities, here called EPR-Bell inequalities or Bell inequalities based on equalities (other examples can be found in [6,7]), whose scaling properties, together with some recent developments like the possibility of achieving almost perfect correlations and the possibility of preparing pairs of particles in hyperentangled states (i.e., entangled in several degrees of freedom [7,8]), make them useful to demonstrate that the violation of EPR’s local realism grows exponentially with the size of the subsystems [9]. An experiment based on this approach has been recently performed in Rome [10]. Here we show how EPR-Bell inequalities can be used to make a loophole-free refutation of EPR’s local realism.

2. TOWARDS A LOOPHOLE-FREE BELL EXPERIMENT

So far, there is no experiment testing the impossibility of EPR’s elements of reality without invoking supplementary assumptions. Some experimental problems make supplementary assumptions necessary. For instance, in most experiments, only a small subset of all the created pairs are actually detected, so the refutation of EPR’s elements of reality is based on the fair sampling assumption (i.e., that the detected pairs are a fair sample...
Therefore, we can define

\[ |\Psi\rangle = \bigotimes_{j=1}^{N} |\psi(j)\rangle, \]  

where

\[ |\psi(j)\rangle = \frac{1}{2}(|00\rangle_1|00\rangle_2 + |01\rangle_1|01\rangle_2 + |10\rangle_1|10\rangle_2 - |11\rangle_1|11\rangle_2). \]

The state \( |\Psi\rangle \) is a state of 4\(N\) qubits in two particles. Consider the following single qubit observables: \( X^{(j)}_k = \sigma_x \otimes I, Y^{(j)}_k = \sigma_y \otimes I, Z^{(j)}_k = \sigma_z \otimes I, x^{(j)}_1 = I \otimes \sigma_x, y^{(j)}_2 = I \otimes \sigma_y \) and \( z^{(j)}_2 = I \otimes \sigma_z \), where \( k \) denotes particle \( k \), \( \sigma_x \) is the Pauli matrix in the \( x \) direction, and \( I \) is the identity matrix in a two-dimensional Hilbert space. For the state \( |\Psi\rangle \), each and every one of these \( 7N \) single qubit observables \( X^{(j)}_1, Y^{(j)}_1, x^{(j)}_1, X^{(j)}_2, Y^{(j)}_2, y^{(j)}_2 \) and \( z^{(j)}_2 \) can be regarded as an EPR element of reality, since it satisfies the following \( 7N \) equalities representing perfect correlations:

\[
\begin{align*}
\langle x^{(j)}_1 X^{(j)}_2 z^{(j)}_2 \rangle &= 1, \quad \langle Y^{(j)}_1 Y^{(j)}_2 z^{(j)}_2 \rangle = -1, \quad \langle x^{(j)}_1 Z^{(j)}_2 y^{(j)}_2 \rangle = 1, \\
\langle x^{(j)}_1 z^{(j)}_2 X^{(j)}_2 \rangle &= 1, \quad \langle Y^{(j)}_1 z^{(j)}_2 Y^{(j)}_2 \rangle = -1, \quad \langle Z^{(j)}_1 y^{(j)}_2 y^{(j)}_2 \rangle = -1, \quad \langle z^{(j)}_1 z^{(j)}_2 \rangle = 1.
\end{align*}
\]

Therefore, we can define

\[
\hat{\beta} = \langle X^{(1)}_1 X^{(1)}_2 \ldots X^{(N-1)}_1 X^{(N-1)}_2 \ldots X^{(N)}_1 X^{(N)}_2 |\psi^{(j)}\rangle \\
- \langle X^{(1)}_1 X^{(1)}_2 \ldots X^{(N-1)}_1 X^{(N-1)}_2 \ldots X^{(N)}_1 X^{(N)}_2 |\psi\rangle, \\
+ \langle X^{(1)}_1 X^{(1)}_2 \ldots X^{(N-1)}_1 X^{(N-1)}_2 \ldots X^{(N)}_1 X^{(N)}_2 |\psi\rangle, \\
- \langle X^{(1)}_1 X^{(1)}_2 \ldots X^{(N-1)}_1 X^{(N-1)}_2 \ldots X^{(N)}_1 X^{(N)}_2 |\psi\rangle, \\
+ \langle Y^{(1)}_1 X^{(1)}_2 \ldots X^{(N-1)}_1 X^{(N-1)}_2 \ldots X^{(N)}_1 X^{(N)}_2 |\psi\rangle, \\
- \langle Y^{(1)}_1 X^{(1)}_2 \ldots X^{(N-1)}_1 X^{(N-1)}_2 \ldots X^{(N)}_1 X^{(N)}_2 |\psi\rangle + \ldots \\
+ \langle Y^{(1)}_1 X^{(1)}_2 \ldots X^{(N-1)}_1 X^{(N-1)}_2 \ldots X^{(N)}_1 X^{(N)}_2 |\psi\rangle,
\]

which contains \( 4^N \) expectation values. For measuring, e.g., \( X^{(1)}_1 x^{(1)}_1 \ldots X^{(N)}_1 x^{(N)}_1 \) on particle 1, we use an analyzer that separates the two possibilities of each of the \( 2N \) qubit observables \( X^{(1)}_1, x^{(1)}_1, \ldots, X^{(N)}_1, x^{(N)}_1 \). This analyzer is backed up by \( 4^N \) particle detectors, one for each of the possible outcomes. Therefore, each particle detection gives the value of \( 2N \) observables. The choice of experiment and the detection of particle 1 are assumed to be random and spacelike separated from those of particle 2.
As can be easily checked, in any EPR-type local realistic theory, $\beta_{\text{EPR}} = 2^N$, while the value predicted by quantum mechanics for the state (1) is $\beta_{\text{QM}} = 4^N$, which violates the EPR bound by an amount which grows as $\beta_{\text{QM}}/\beta_{\text{EPR}} = 2^N$, assuming perfect states and measurements. The remarkable point is that this exponentially growing-with-size violation of EPR’s local realism can be demonstrated by actual experiments if we use two-particle hyper-entangled states. In practice, we do not have perfect correlations but

$$\langle X_1^{(1)} x_1^{(1)} \ldots X_1^{(N)} x_1^{(N)} Y_2^{(1)} y_2^{(1)} \ldots Y_2^{(N)} y_2^{(N)} \rangle = 1 - \epsilon,$$

where $\epsilon \approx 0.15$ [10]. In a worst-case scenario, each of the terms in $\hat{\beta}$ is affected by a similar error. Since the number of terms in $\hat{\beta}$ is $4^N$, then we should take into account that our value for $\beta_{\text{EPR}}$ could be increased to $\beta'_{\text{EPR}} \approx 2^N + 4^N 0.15$. Also, we must take into account the imperfection in the preparation of the state which, in practice, is not $|\Psi\rangle$, but $\rho = p|\Psi\rangle\langle\Psi| + (1 - p)\rho'$, with $p \approx 0.98$, and the specific form of the term $\rho'$ depends on the physical procedure used to prepare and distribute the state. Therefore, the expected experimental value of $\hat{\beta}$ is $\beta'_{\text{QM}} \approx 0.98 \times 4^N + 0.02$. The interesting point is that $\beta'_{\text{QM}}$ still provides a growing-with-size violation of the inequality $\hat{\beta} \leq \beta'_{\text{EPR}}$. In Table 1 we have calculated the minimum detection efficiencies required for a loophole-free experiment based on this approach. For more details, see [15].

Table 1: Minimum detection efficiencies required for a loophole-free Bell experiment based on: (a) the EPR-Bell inequality which is a “product” of CHSH inequalities and hyper-entangled states which are the product of Bell states, like in the Rome experiment [10]; (b) the EPR-Bell inequality described in this paper and the states (1). $n$ is the number of qubits encoded in each photon. The lower bound for the efficiency is calculated assuming that the EPR-Bell inequality is valid for all pairs, even if we do not have perfect correlations. The upper bound is calculated assuming that the EPR-Bell inequality is only valid for the fraction of pairs for which we have perfect correlations (0.85) and the other fraction (0.15) conspires in order to simulate the quantum prediction.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\eta_{(a)}$</th>
<th>$\eta_{(b)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.83 - 0.96</td>
<td>...</td>
</tr>
<tr>
<td>2</td>
<td>0.67 - 0.89</td>
<td>0.67 - 0.79</td>
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<tr>
<td>3</td>
<td>0.52 - 0.87</td>
<td>...</td>
</tr>
<tr>
<td>4</td>
<td>0.40 - 0.92</td>
<td>0.40 - 0.57</td>
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<td>5</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>6</td>
<td>...</td>
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</tr>
<tr>
<td>7</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>8</td>
<td>...</td>
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<tr>
<td>9</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>10</td>
<td>...</td>
<td>0.06 - 0.31</td>
</tr>
</tbody>
</table>

These results suggest that EPR-Bell inequalities derived for some specific hyper-entangled states can be useful to experimentally refute elements of reality, without requiring the fair sampling assumption.
ACKNOWLEDGMENTS

The author acknowledges support from Project No. FIS2005-07689.

REFERENCES


