Kochen-Specker Meets Experiments

Adán Cabello

Departamento de Física Aplicada II, Universidad de Sevilla, E-41012 Sevilla, Spain

Abstract. The assumption of non-contextuality is, in some scenarios, not only physically "plausible," but unavoidable for any realistic theory without instantaneous actions. Any realistic non-contextual theory must satisfy some inequalities for the correlations of compatible (commensurable) measurements, which are violated by any quantum state. Actual experiments can reveal this state-independent violation. We discuss the requirements and the advantages of these experiments.

Keywords: Kochen-Specker theorem, Non-contextuality, Quantum contextuality, Bell’s theorem.

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INTRODUCTION

Unperformed experiments meet actual experiments

The question is whether the outcomes of any possible interaction between an individual piece of the universe (e.g., a physical system under observation) and other piece of the universe (e.g., a measurement device) are predetermined. On the answer to this question relies the possibility of deeper levels of description in which the outcomes of the interactions are determined by the “properties” of the interacting parts. A negative answer would point out a limit to any description of the universe, and it would be in itself very informative about the structure of the universe. The fact that such a question can be actually tested in experiments is one of the most fascinating subjects in the recent history of science.

The possibility of an experimental test is opened when one makes extra assumptions on these (determined) outcomes, and confront the predictions of these more detailed descriptions with those of quantum mechanics (QM). The assumption we want to test is:

(i) “Realism: measurement outcomes of nonperformed measurements can be introduced alongside of those of the actually performed measurements” [1].

In order to make it testable, we need to be more specific; one option would be to make the following additional assumptions:

(ii) “Locality: the measurement outcome at Alice’s station does not depend on Bob’s choice of setting” [1], because Alice’s outcome and Bob’s choice are space-like separated and therefore, according to special relativity, Bob’s choice cannot influence Alice’s outcome.

(iii) “Freedom: Alice and Bob can perform either measurement” [1].

Assumptions (i), (ii), and (iii) are collectively known as local realism.

Alice meets Bob

This is not the only way to test realism. This paper deals with another possibility, which consists of replacing (ii) by (ii’) Non-contextuality: the outcome of Alice’s measurement does not depend on Bob’s (previous, simultaneous, or future) compatible (i.e., commensurable) measurement.

Alice and Bob are not necessarily in two distant stations anymore; both can be in the same location and interact with the same individual system (e.g., Alice first measures the system and then Bob performs an additional measurement).

Locality is a particular example of non-contextuality since two observables on different particles are compatible. However, two observables on the same particle can also be compatible. Locality is considered a physically "plausible"
assumption. In the next section we will argue that there are also reasons to consider non-contextuality a plausible assumption. We will then describe how to experimentally test assumptions (i), (ii’), and (iii), which are collectively referred to as non-contextual realism. Finally, in the last sections, we will summarize the benefits of assuming non-contextuality instead of locality.

IS NON-CONTEXTUALITY A PHYSICALLY PLAUSIBLE ASSUMPTION?

Bell meets Kochen-Specker

The conflict between local realism and QM is known as Bell’s theorem. It was discovered by Bell in [2] and then refined by Clauser, Horne, Shimony, and Holt (CHSH) in [3]. The properties of special quantum states are necessary to reveal the conflict.

The conflict between non-contextual realism and QM is known as the Kochen-Specker (KS) theorem. The conflict was first pointed out by Specker in [4], and proved by Kochen and Specker in [5]. This proof used structures discussed in [6, 7]. A physical system in which the number of compatible questions exceeds 2 is necessary to reveal the conflict. However, the conflict occurs for any quantum state.

According to [8], in 1963 Jauch pointed out to Bell that a corollary of Gleason’s theorem [9] implied a conflict between non-contextual realism and QM. In [10], Bell proved that it was so. Bell was not aware of the work of KS until [8], where Bell cites the famous KS paper [5] for the first time, and where he adds the following note in proof:

“I am sorry to have missed, before writing the above, an earlier paper by E. Specker [4]. It announced already what I have called the Gleason–Jauch result. Specker did not know of the work of Gleason, but mentioned rather the possibility of an ‘elementary geometrical argument’—presumably of the kind I myself gave later [10] as a preliminary to criticism of the axioms” [8].

Bell did not like the assumptions of the KS theorem. More specifically, he disliked non-contextuality.

Plausibility and implausibility in quantum mechanics

Bell’s criticism to non-contextuality can be summarized as follows:

“It was tacitly assumed that measurement of an observable must yield the same value independently of what other measurements may be made simultaneously. Thus as well as A say, one might measure either B or C, where B and C are orthogonal to [i.e., compatible with] A but not to one another. These different possibilities requires different experimental arrangements; there is no a priori reason to believe that the results for A should be the same. The result of an observation may reasonably depend not only on the state of the system (including hidden variables) but also on the complete disposition of the apparatus” [10].

Bell was wrong on one point. The conflict between non-contextual realism and QM does not depend on the extra assumption that “experimental results [are] regarded as ‘measurements’ of preexisting properties of the ‘system’ alone” [8], the conflict persists even if the experimental results “depend on the state of the system (including hidden variables) and the complete disposition of the apparatus” [10] or are “regarded as the joint product of the ‘system’ and ‘apparatus,’ the complete experimental set-up” [8].

However, Bell also pointed out the weakness of the KS theorem. Is there any physical reason to believe that the result of A should be the same when A is measured alone, or together with B, or together with C? Bell thought that there is no such reason. Sixteen years later, Bell still insisted in this point:

“C commute [i.e., is compatible] with A (…), but not with B. And the extra assumption is this: the result of ‘measuring’ A is independent of which complementary set, B or C, is ‘measured’ at the same time. (…) Even though the two sets of operators have A in common, the eigenvectors are different (…). There is nothing unacceptable, or even surprising, about this. (…) We are doing a different experiment when we arrange to ‘measure’ C rather than B along with A. The apparent freedom of the Gleason–Jauch argument from implausible assumptions about incompatible ‘observables’ is illusory” [8].
Is non-contextuality a physically plausible assumption? For any realistic theory with no “instantaneous” actions, non-contextuality can be an unavoidable assumption, at least in some situations.

**Sequential versus simultaneous measurements**

“Perhaps it [non-contextuality] seems particularly plausible when the commuting [i.e., compatible] ‘observables’ involved are ‘measured’ at the same time” [8].

Non-contextuality may seem to be more plausible when \( A \) and \( B \) are measured “simultaneously,” if we believe that the perturbation caused by one measurement does not have time to modify the result of the other measurement. However, the price for a simultaneous measurement of \( A \) and \( B \) is that it requires not only a different “experimental arrangement” than the one required for a simultaneous measurement of \( A \) and \( C \), but an entirely different measuring device. One could argue that there is nothing wrong in using two different devices to measure the same observable because, according to QM, both devices will give exactly the same statistical distribution of results. However, such a digression is unnecessary, since one can make a more transparent argument using the *same* device to measure \( A \) in both cases. Exactly the same way as, in a Bell experiment, Alice always uses the same device to measure \( A \), regardless of whether Bob measures \( B \) or \( C \).

Suppose that, instead of measuring \( A \) and \( B \) simultaneously, they are measured sequentially (i.e., first Alice measures \( A \) and then Bob measures \( B \)). For that purpose, they need a device that only measures \( A \) (or \( B \)) and nothing else, so that one can guarantee that the result of a subsequent measurement of any compatible observable is not disturbed by the previous measurement of \( A \) (or \( B \)).

If \( A \) and \( B \) are dichotomic observables, then each of them must be measured by a device with one input and two possible outputs. In this case, a sequential measurement of \( A \) and \( B \) consists of an arrangement in which the individual systems enter the device for measuring \( A \) through the input, interact with the device for measuring \( A \), and this interaction forces each individual system to choose one of two possible outputs. If we would place a detector in each of these outputs, we would always have a single detection. However, instead of placing two detectors, one in each of \( A \)’s outputs, we place two identical devices for measuring \( B \), one after each of the two outputs of \( A \). Then, an individual system passing through the whole arrangement will produce a single detection in one of the 4 detectors. This detection gives the results for \( A \) and \( B \). The scheme is illustrated in Fig. 1.

**FIGURE 1.** Scheme of a sequential measurements of two compatible dichotomic observables \( A \) and \( B \). The possible outcomes of each device (box) are labeled “+” and “−.” The two devices for measuring \( B \) are identical. A detection in the detector labeled “\( A + B \)−” means that the result of measuring \( A \) is “+” and the result of measuring \( B \) is “−.”

The operational criterion to decide whether two devices actually measure two compatible observables \( A \) and \( B \) is the following: If, for any individual system, in any state, one first measures \( A \), and next \( B \), and then \( A \) again, as in Fig. 2, then one will never obtain any detection in the outputs \( A + B \)−, \( A + B \)−, \( A + B \)−, \( A + B \)−, but \( A + B \)−. Moreover, on the same individual system, one can measure \( A \) again or make some other more compatible measurements and then measure \( A \) again. If the devices are compatible, one will always obtain the same result for \( A \).

This gives us a physical reason to believe that the result of measuring \( A \) the second time has not been disturbed by the prior measurement of \( B \). However, Bell would probably argue that the result of measuring \( B \) can be reasonably disturbed by the prior measurement of \( A \).
FIGURE 2. Scheme of a test of compatibility. If A and B are compatible, then the detectors “A + B + A–,” “A + B – A–,” “A – B + A+,” and “A – B – A+” never detect anything.

Non-contextuality meets predictability

The uncertainty principle does not prevent an individual system from having sharply defined joint values of compatible observables. We can prepare an individual system and know the results of measuring a set of compatible observables before actually performing the measurements. These results are the same, regardless of the order in which we perform the measurements. This suggests that this individual system possesses definite responses to all these interactions. However, this is valid only for individual systems prepared in a simultaneous eigenstate of all the compatible observables measured. If one sequentially measures a different set of compatible observables, it might happen that the result of the second measurement is altered by the first measurement.

However, for any individual system, in any state, when we repeatedly measure compatible observables (as in Fig. 2), we always obtain the same result for the same observable, regardless of the order and the number of times we measure each observable. Therefore, the result of the second observable B may be altered by the first measurement of the first observable A, but if we next measure any compatible observable C (including, again A), then any subsequent measurement of B will give the same result. However, it still might happen that the result of B is “fixed” by the first measurement of A.

In any realistic theory, the fact that a specific observable must have a predefined response is assumed a priori. The fact that one system cannot have responses to any interaction is not surprising by itself. If you are in a room with two windows and someone asks you to jump through one of them, without giving you the option of refusing to do so, your final choice is probably not predefined by your previous “state” but more likely is a last second and apparently random response to an unexpected request.

However, what if someone could predict with certainty (i.e., with probability 1) your response even before you were actually asked to jump through a window? What if someone could predict with certainty your response to any possible request? Then you would probably be tempted to admit that there must be something in your state or in the state of the measuring device that determines your response.

Any individual system which is maximally entangled with a distant ancillary system is locally (i.e., if we ignore the ancilla) in a maximally mixed state (i.e., with no information). However, someone can predict with certainty the result of any measurement from the result of the same measurement on the ancilla. In this scenario, why must any defender of realism believe that the result of A is the same when A is measured alone, when A is measured with B, and when A is measured with C? Because the predicted result is the same in every context. The measurement on the ancilla can be made space-like separated from the measurement on the system, so no mutual influence can take place (according to special relativity). Here we are invoking Einstein, Podolsky, and Rosen’s (EPR’s) sufficient condition to guarantee that a result is “real” [11]. The observation that the KS theorem acquires a new perspective when applied to systems which are maximally entangled with other systems and when the EPR criterion is invoked was first made by Heywood and Redhead [12].

To sum up, the assumption of non-contextuality of the results of sequential compatible measurements is particularly plausible for individual systems which are maximally entangled with an ancilla. For these systems, non-contextuality seems unavoidable for any realistic theory without instantaneous actions.
KOCHEN-SPECKER MEETS INEQUALITIES

Bell’s theorem meets competence

“One reason the Bell-KS theorem is the less celebrated of the two [theorems of impossibility of hidden-variables, the other is Bell’s theorem] is that the assumptions made by the hidden-variables theories it prohibits can only be formulated within the formal structure of quantum mechanics. One cannot describe the Bell-KS theorem to a general audience, in terms of a collection of black-box gedanken experiments, the only role of quantum mechanics being to provide gedanken results, which all by themselves imply that at least one of those experiments could not have been revealing a preexisting outcome. (...) A less edifying reason for the greater fame of Bell’s Theorem among physicists is that its proof is utterly transparent” [30].

There is a third fundamental reason why the KS theorem is not so famous among physicists. It is because it was thought that it could not be experimentally tested. There was a time when it was thought that “the whole notion of an experimental test of KS misses the point” [13]. Then, it was believed that “finite-precision measurement nullifies the Kochen-Specker theorem” [14]. Fortunately, the KS theorem survived these debates [15–23] and there are experiments and proposals to prove “quantum non-contextuality” with photons [24], neutrons [25], and superconducting qubits [26].

However, none of these experiments and proposals captures the full power of the KS theorem. In particular, none of them tests of one of the main features of the KS theorem: The conflict is independent of the state of the individual system. The proposal in the next section captures the essence of the KS theorem and provides a feasible way to experimentally test it.

5 minus 1 equals 6

Consider a physical system in which the number of compatible questions is 4 (e.g., two qubits or a single spin-3/2 particle). Suppose that \( A, B, C, a, b, c, \alpha, \beta, \) and \( \gamma \) are dichotomic observables (interactions with a device that can only have two possible outputs) and that we label the two possible results \(-1\) and \(+1\). Suppose that we assume that every individual system has a definite response to each of these interactions. Suppose that this response is the same regardless of which other compatible interactions occur on the same individual system. Then, it can be proved that if 3 compatible observables \( A, B, \) and \( C \) are measured on all the individual systems of a subsense, another 3 compatible observables \( A, a, \) and \( \alpha \) are measured on all the individual systems of a different subsense, etc., so that incompatible observables are never measured on the same individual system, then the averages of the products of the results of these measurements must satisfy the following inequality:

\[
\langle ABC \rangle + \langle abc \rangle + \langle \alpha \beta \gamma \rangle + \langle Aa \alpha \rangle + \langle Bb \beta \rangle - \langle Cc \gamma \rangle \leq 4,
\]

where, e.g., \( \langle ABC \rangle \) is the average of the products of the results \((-1, +1)\) of measuring \( A, B, \) and \( C \), where the average is taken over all the individual systems on which \( A, B, \) and \( C \) have been measured.

The conflict with QM appears when we consider, e.g., the following observables of a two-qubit system:

\[
A = \sigma_z^{(1)}, \quad B = \sigma_z^{(2)}, \quad C = \sigma_z^{(1)} \otimes \sigma_z^{(2)},
\]

\[
a = \sigma_z^{(1)}, \quad b = \sigma_z^{(1)}, \quad c = \sigma_z^{(1)} \otimes \sigma_z^{(2)},
\]

\[
\alpha = \sigma_x^{(1)} \otimes \sigma_x^{(2)}, \quad \beta = \sigma_x^{(1)} \otimes \sigma_x^{(2)}, \quad \gamma = \sigma_x^{(1)} \otimes \sigma_x^{(2)},
\]

where \( \sigma_z \) denotes the Pauli matrix \( Z \) of qubit 1. Then, according to QM, for any two-qubit state, the left-hand side of (1) must be 6, since \( ABC = abc = \alpha \beta \gamma = Aa \alpha = Bb \beta = Cc \gamma = \mathbb{I} \), where \( \mathbb{I} \) denotes the identity. QM violates inequality (1) for any two-qubit state.

This inequality was first introduced in [27] and exploits the properties of a proof of the KS theorem introduced by Peres and Mermin [28–31]. Other inequalities violated by any state are also introduced in [27]. A method to convert any proof of the KS theorem into an inequality between the correlations of compatible measurements which is violated by any state is presented in [32].
Testing the inequality

Observing the state-independent violation predicted by QM in an actual experiment is a major challenge. For that purpose, inequality (1) seems to be particularly suitable, since most of the requirements for the experiment have already been addressed, at least in the case where the physical system is a two-qubit system consisting of the spatial and the spin components of a single neutron [25]. Other possibility is to use the polarization and path degrees of freedom of a single photon [24].

![Diagram of quantum states](image)

**FIGURE 3.** Scheme of a sequential measurement of 3 compatible observables A, B, and C. If A, B, and C are those defined in (2), then the prediction of QM is that the detectors “A + B + C−,” “A + B − C+,” “A − B + C+,” and “A − B − C−” never detect anything.

To test inequality (1), one has to prepare an individual system in a specific two-qubit quantum state (e.g., a maximally entangled state), measure, e.g., A, B, and C (a sequential measurement is illustrated in Fig. 3). The prediction of QM is independent of the order of the measurements (A−B−C, as in Fig. 3, A−C−B, B−A−C, B−C−A, C−A−B, or C−B−A). One can test this prediction by arranging the devices so that each of the 6 possible sequences occurs with the same frequency. Then, one has to prepare another individual system in the same state and measure, e.g., A, a, and α, and repeat these measurements several times, until one has obtained enough data to calculate the 6 mean values in (1) for this state.

Then, one has to repeat the experiment with different states (e.g., a partially entangled state, a product state, different Werner states, a maximally mixed state, and a maximally mixed state which is maximally entangled with an external ancilla). The violation predicted by QM is the same for every state. The case of the maximally mixed state which is maximally entangled with an external ancilla is particularly interesting, since in there any of the results of the measurement of any of the 9 observables (2) can be predicted with certainty from the result of the measurement of the same observable on the ancilla (this can also be tested), which provides a physical motivation for the assumption of non-contextuality.

Measures and loopholes

There are two main measures that quantify how good is the violation of a Bell inequality. Here we calculate them for the violation of inequality (1).

- **Resistance to noise.** The first measure is the minimum “visibility” $\mathcal{V}_{\text{crit}}$ required to violate the inequality if, instead of the state $|\psi\rangle$ that gives the maximum violation, the state on which the inequality is tested is $\rho = \mathcal{V} |\psi\rangle \langle \psi| + \frac{(1-\mathcal{V})I}{2}$, where $I$ is the identity matrix in the Hilbert space of the whole system. $\mathcal{V}_{\text{crit}}$ measures the robustness of the violation to the state’s imperfections in the preparation and distribution. For instance, for the CHSH Bell inequality [3], the violation occurs only if the actual visibility is higher than $\mathcal{V}_{\text{crit}} = 1/\sqrt{2} \approx 0.71$. For a given $\mathcal{V}$, the degree of violation $\mathcal{D}$ (i.e., the ratio between the observed result and the bound of the inequality) is $\mathcal{D} = \sqrt{2\mathcal{V}}$. For inequality (1), the violation occurs for any visibility (i.e., $\mathcal{V}_{\text{crit}} = 0$), and $\mathcal{D} = 3/2$ independently of $\mathcal{V}$.

- **Robustness against imperfect detectors.** The second measure is the minimum overall detection efficiency $\eta_{\text{crit}}$ (i.e., the ratio detected/measured individual systems) required to avoid the possibility of the violation being simulated by a hidden variable model in which each individual system has “be go undetected” as a possible response to the measurements [33]. For instance, for the CHSH Bell inequality, local hidden variable models can be excluded only if the actual overall detection efficiency is higher than $\eta_{\text{crit}} = 2/(1 + \sqrt{2\mathcal{V}})$ [34, 35] (i.e., $\eta_{\text{crit}} = 2(\sqrt{2} - 1) \approx 0.83$ for $\mathcal{V} = 1$).
In order to calculate, \( \eta_{\text{crit}} \) for the inequality (1), we consider hidden variable models in which the responses “give the outcome +” and the “give the outcome −” are non-contextual, but the responses “be undetected” are contextual. The reason for this different treatment is that, while for a detected system one can potentially predict the outcome (e.g., from the result of a distant measurement with perfect detectors on an ancilla), so there is a physical reason to believe that this outcome is non-contextual; such a possibility does not exist for a non-detected system. Therefore, we consider models in which a single system would be go undetected only when \( A, B, \) and \( C \) are measured, although it would be detected in any of the other 5 measurements, even though some of them involve \( A, B, \) or \( C \). If we denote by \( P(\vert ABC \rangle = 1) \) the probability that an individual system is detected after a sequential measurement of \( A, B, \) and \( C \) then, in order to reproduce the behavior of actual detectors, the hidden variable model must satisfy \( P(\vert ABC \rangle = 1) = P(\vert abc \rangle = 1) = P(\vert \alpha \beta \gamma \rangle = 1) = P(\vert Bb\beta \rangle = 1) = P(\vert Cc\gamma \rangle = 1) = P(\vert \alpha \beta \gamma \rangle = 1) = \eta \), where \( \eta \) is the overall detection efficiency simulated by the model. In order to construct a model capable of simulating the maximum \( \eta \), it is enough to consider the model in which each individual system would be go undetected only when one of the 6 measurements is performed (but would be detected when any of the other 5 measurements is performed) and each of the 6 possible hidden variable states occurs with equal frequency. For this model, \( \eta = 5/6 \approx 0.83 \), which is \( \eta_{\text{crit}} \) for the inequality (1). This value is independent of \( \gamma \). A similar argument leads us to the conclusion that, for the inequalities in [27, 32], \( \eta_{\text{crit}} = (n - 1)/n \), where \( n \) is the number of different (mutually incompatible) experiments needed to test the inequality. The minimum possible \( \eta_{\text{crit}} \) is 4/5 (which is the \( \eta_{\text{crit}} \) for the third inequality in [27]).

Bell experiments are affected by two main loopholes. Here we discuss how these loopholes affect experimental violations of inequality (1).

- **Locality loophole.** Locality can be invoked only when Alice’s (Bob’s) outcome and Bob’s (Alice’s) choice are space-like separated. Otherwise, one observer’s measurement choice can affect the result of the other observer’s measurement [36]. Ironically, this requirement is not satisfied in the vast majority of Bell experiments (two exceptions are [37, 38]). Inequality (1) is not based on locality. Therefore, any test of (1) is not affected by this loophole.

- **Detection loophole.** As stated above, in order to avoid the possibility of the violation being simulated by a hidden variable model in which each individual system has “be go undetected” as a possible response, the detector must be good enough, as in a Bell experiment. However, the detection loophole is much easier to close in a test of inequality (1) than in a Bell experiment, since no spatial separation is required and single or pairs of massive particles can be used [39, 40].

Since a loophole-free test requires \( \eta > 0.83 \), the best option for an actual loophole-free experiment seems to be two ions in the same trap [39], or a single ion with four levels.

**CONCLUSION**

I prefer to think that QM suggests that the outcomes of the interactions are not predetermined. The state of the observed system plus the measuring device does not have responses to some interactions before the interactions take place. These responses are forced into existence by the act of interaction itself. It makes no sense to introduce measurement outcomes of unperformed measurements alongside those of actually performed measurements, even if they can be potentially predicted with certainty from the results of distant measurements: “Unperformed experiments have no results” [41]. Realism makes no sense.

In this conclusion, the role of the choice between locality and non-contextuality is secondary. We just need one of them to be more precise in order to reveal a conflict. I would rather not base the conclusion on the properties of some special quantum states. This could suggest that realism can still make sense for certain states. This could suggest that “classical states” and quantum states can coexist in a description of the universe. In Bell’s theorem, the use of entangled states hides a much more general conflict. In this sense, entanglement is not “the characteristic trait of quantum mechanics, the one that enforces its entire departure from classical lines of thought” [42].

I prefer to base the conclusion on a clever election of interactions (suggested by QM) which reveals that the conflict occurs for any state. From this perspective, there are no “classical states.” There is no “classical limit” beyond the case when the violation of the inequality (1) vanishes because of the lack of perfection of the devices. There are no “classical interactions,” just careless measurements.
Revealing the conflict does not require considering the interactions of a composite system, but just the interactions of a piece of the universe that admits more than two compatible experiments. We can forget the locality loophole that affects Bell experiments. The detection loophole is much easier to close than in Bell experiments because no spatial separation is required. Even with simple systems (e.g., admitting 4 compatible observables), QM’s predictions are sufficiently different to those of non-contextual realistic theories to allow us to experimentally exclude non-contextual realism for any state.

ACKNOWLEDGMENTS


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